Lecture 10

Dynamic Asset Pricing Models - I

Consumption-CAPM

- We like the CAPM and the APT because they both capture risk and return, but are their related to our more fundamental needs: consumption of goods.
- What do we mean by "The equity premium puzzle is too high"?
- We will work out a "simple" model where assets are priced explicitly relative to our utility from consumption.
- This explicit model will generate a familiar stochastic discount factor pricing relation.
- A structural model like this may answer the question: What do we mean by "The equity premium puzzle is too high"?

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• The model starts with a representative agent (average Joe), who invests today to consume tomorrow. Average Joe's utility function:

$$U(c_t,c_{t+1})$$

c_t: consumption at date t.

• u(.) is increasing –i.e., u'(.)>0- and concave –i.e., u''(.)<0.

• u(.) should display aversion to risk and to inter-temporal substitution: Average Joe prefers a consumption stream that is steady over time and across states of nature.

• Average Joe's two period problem: Maximize expected utility. Average Joe's can save by buying an asset, x_t, with an uncertain payoff –he can also save by buying a risk-free asset.

$$\max_{\varphi} \{U(c_t,c_{t+1})= \Sigma_s \pi_s u(c_t,c_{t+1}(s)\} \quad c_t € R, c_{t+1} € R^s \qquad \text{s.t.}$$

$$c_t = a_t - p_t\varphi$$

$$c_{t+1} = a_{t+1} + x_{t+1}\varphi$$

- a_t: endowment of the individual at time t.
- p_t: price of asset at time t.
- φ shares purchased at price p_t.

• With this setup, we want to find p_t –it will be a function.
• This is a structural model –an equilibrium model.

• FOC:

$$p_t = E_t[(u'_{t+1}/ u'_t)\, x_{t+1}] = E_t[m_{t+1}\, x_{t+1}] \qquad \text{(Euler's equation)}$$

where $m_{t+1} = (u'_{t+1}/ u'_t)$ = intertemporal MRS. $m_{t+1} >0$, from the assumption about u(.).

• Back to the Euler's equation:

$$p_t = E_t[m_{t+1} x_{t+1}]$$

• m_{t+1} is called the *stochastic discount factor* (SDF). The value of the asset, p_t, is equal to the expected m-discounted future payoff.

• m_{t+1} is also called the *pricing kernel*.

•Any function that satisfies the Euler's equation is an admissible SDF.

• Economists derive SDFs from generic (conditional) factor models:

If asset returns are given by: $R_{i,t} = \mu_{t-1} + \boldsymbol{\beta}_{\mathbf{t-1}} \mathbf{F}_t + \varepsilon_t$

There exist a_{t-1} and B_{t-1}, such that $m_t = a_{t-1} + \mathbf{B}_{\mathbf{t-1}} \mathbf{F}_t$ is an admissible SDF

That is, $p_{i,t} = E_t[m_{t+1} x_{i,t+1}]$ for all assets.

Aside: Where does the SDF come from?

- Suppose there are two equally likely states: S=2, $\pi_{s=1}$= ½.
- Average Joe comes with:
  – endowment: 1 in date t, (2,1) in date t+1
  – utility function $E[U(c_t,c_{t+1})] = \Sigma_s \pi_s\{\ln c_t + \ln c_{t+1}(s)\}$
    –i.e., $u(c_t,c_{t+1}(s)) = \ln c_t + \ln c_{t+1}(s)$ (additive time separable)

- $m_{t+1} = (u'_{t+1}/ u'_t) = \partial_{t+1}u(1;2,1)/E[\partial_t u(1;2,1)]$
  $= (c_t/c_{t+1}(1), c_t/c_{t+1}(2))=(1/2, 1/1)$
  $\Rightarrow m_{t+1} = (½,1)$ and $E_t[m_{t+1}] = ¾$
- Low consumption states are high "m-states"
- Euler's condition can be derived from no-arbitrage. Thus, risk-neutral probabilities combine true probabilities and marginal utilities.

m_{t+1} is a function of consumption. This is a consumption-based pricing model:

$$p_t = E_t[m_{t+1}\ x_{t+1}] \qquad \text{(CC.0)}$$

or

$$p_t = E_t[x_{t+1}]\ E_t[m_{t+1}] + Cov_t[x_{t+1},\ m_{t+1}]$$

(CC.0) must hold for any asset. Suppose there is a risk-free asset:

$$1 = E_t[(1+r_f)\ m_{t+1}] \qquad \Rightarrow\ 1+r_f = 1/E_t[m_{t+1}]$$

Note that the risk-free rate depends on m_{t+1}. (If no risk-free rate, r_f should be interpreted as Black's (1972) zero-beta portfolio.)

• Then,

$$p_t = 1/(1+r_f)\ E_t[x_{t+1}] + Cov_t[x_{t+1},\ m_{t+1}]$$

Interpretation: Price = Expected PV + Risk adjustment

=> Positive correlation with SDF (a function of consumption) adds value

• Divide (CC.0) by p_t to get an expression in terms of returns:

$$1 = E_t[m_{t+1}\ x_{t+1/}\ p_t\ ] = E_t[m_{t+1}(1+R_{t+1})] = E_t[m_{t+1}\ z_{t+1})]$$

where $z_{t+} = (1+R_{t+1})$. Using the definition of covariance:

$$1 = E_t[z_{t+1}]\ E_t[m_{t+1}] + Cov_t[z_{t+1},\ m_{t+1}] \qquad \text{(CC.1)}$$

For a risk-free asset: $1/(z_f) = E_t[m_{t+1}]$

Substituting into (CC.1) and solving for z_f:,

$$z_f = E_t[z_{t+1}] + Cov_t[z_{t+1},\ m_{t+1}]\ z_f\ .$$

or

$$E_t[z_{t+1} - z_f] = -\ Cov_t[z_{t+1},\ m_{t+1}]\ z_f\ .$$

This is the basis of the C-CAPM: excess return or risk premium is determined by its the covariance with the SDF –i.e., a function of consumption. To estimate the model we need m_{t+1}.

$$E_t[z_{t+1} - z_f] = - Cov_t[z_{t+1}, m_{t+1}]\, z_f .$$

Again, to estimate the model we need m_{t+1}. Depending on the assumption to derive m_{t+1}, there are many variants of the C-CAPM.

- A popular C-CAPM version:

(1) Assume a time-additive utility function

$$u(c_t) + \beta E_t\, u(c_{t+1})$$
$$m_{t+1} = \beta\, u'(c_{t+1})/ u'(c_t)$$

β: subjective discount factor, usually $\beta<1$. (In behavioral finance: $\beta\geq 1$.).

(2) Assume a power function for u(.):

$$u(c_t) = [1/(1-\gamma)]\, c_t^{1-\gamma} \quad (\gamma \text{ captures risk aversion}).$$

Relative AP = RRA = $- u''(.)/u'(.)\, c_t = \gamma$ (constant relative risk aversion)

$$m_{t+1} = \beta(c_{t+1}/c_t)^{-\gamma}$$

Note: Both are not trivial assumptions.

- In particular, the power utility function has important implications:
  - It is scale-invariant: risk premia do not change over time as aggregate wealth and the scale of the economy increases. Good property.
  - If investors have the same power utility function, even with different endowments, it aggregates well. Good result for Average Joe.
  - The elasticity of intertemporal consumption (EIS) is the inverse of γ. There is no economic reason to expect this link. Bad property.

Epstein and Zin (1989, 1991), and Weil (1989) develop a more flexible version of the power utility model, breaking the link between the EIS and γ.

• Back to the model, we substitute the power utility in the FOC:

$$1 = E_t[m_{t+1}\,(1+R_{t+1})] = E_t[\beta(c_{t+1}/c_t)^{-\gamma}\ (1+R_{t+1})] \qquad \text{(CC.2)}$$

Note: If we have data on R_{t+1} and on c_t , we can estimate β and γ. But, the relation is non-linear.

• We need an additional assumption to deal with uncertainty –i.e., the conditional expectation: Log-normality for X_t=$m_{t+1}\,(1+R_{t+1})$.

Recall: If X_t is conditionally lognormally distributed, it has the convenient property:

$$\ln E_t[X_t] = E_t[\ln(X_t)] + (1/2)\ Var_t[\ln(X_t)] \qquad \text{(Assume } Var_t[\ln(X_t)]=\sigma_x^2\text{).}$$

(Thus, we assume joint conditional lognormality and homoscedasticity of asset returns and consumption. These are non-trivial assumptions.)

• Recall (CC.2): $1= E_t[\beta(c_{t+1}/c_t)^{-\gamma}\ (1+R_{t+1})]$

Taking logs:

$$0 = \ln E_t[\beta(c_{t+1}/c_t)^{-\gamma}\ (1+R_{t+1})]$$

$$= E_t[\ln(\ \beta(c_{t+1}/c_t)^{-\gamma}\ (1+R_{t+1}))] + (1/2)\ Var_t[\ln(\ \beta(c_{t+1}/c_t)^{-\gamma}\ (1+R_{t+1}))\ ]$$

$$= \ln\beta - \gamma\ E_t[\ln(c_{t+1}) - \ln(c_t)] + E_t[r_{t+1}] + (1/2)\ [\sigma_r^2+\gamma^2\ \sigma_\Delta^2 - 2\gamma\sigma_{r\Delta}] \quad \text{(CC.3)}$$

where $r_{t+1}= \ln(1+R_{t+1})$

$\sigma_r^2 = Var[\ln(1+R_{t+1})] = Var(r_{t+1})$

$\sigma_\Delta^2 = Var[\ln(c_{t+1}) - \ln(c_t)]$

$\sigma_{r\Delta} = Cov[\ln(c_{t+1}) - \ln(c_t),\ r_{t+1}]$

• (CC.3) is valid for any asset.
  - In particular, for the risk-free asset:

$$0 = \ln\beta - \gamma\ E_t[\ln(c_{t+1}) - \ln(c_t)] + r_f + (1/2)\ \gamma^2\ \sigma_\Delta^2$$

or

$$r_f = -\ln\beta + \gamma\ E_t[\ln(c_{t+1}) - \ln(c_t)] - (1/2)\ \gamma^2\ \sigma_\Delta^2$$

- For a risky asset:

$$E_t[r_{t+1}] = -\ln\beta + \gamma\, E_t[\ln(c_{t+1}) - \ln(c_t)] - (1/2)\,[\sigma_r^2 + \gamma^2\, \sigma_\Delta^2 - 2\gamma\sigma_{r\Delta}]$$

• Now, we can calculate excess returns for a risky asset:

$$E_t[r_{t+1}] - r_f = \gamma\sigma_{r\Delta} - \sigma_r^2/2 \qquad \text{(CC.4)}$$

The excess return is a (linear) function of the covariance of the asset with consumption growth.

• (CC.4) is a Consumption CAPM (C-CAPM) version.
  - There is no need to estimate a market portfolio. We only need an estimate of consumption growth to estimate this model.
  - The coefficient γ has a very nice interpretation: It measures risk aversion.
  - The C-CAPM (with the added log-linearity restrictions). It is easy to test using linear regressions.

Classic references: Lucas (1978), Breeden (1979), Hansen and Singleton (1982).

Testing C-CAPM: GMM

• GMM can naturally be applied in the C-CAPM. The Euler's equation, gives us a starting point for a moment condition:

$$0 = E_t[\beta(u'(c_{t+1})/\, u'(c_t))\, (1+R_{t+1}) - 1].$$

Let $\mathbf{Z}_t$ be a set of L ($L \geq K$) instruments, available at time t. Then, for each asset i:

$$E_t[Z_j\ \{\beta(c_{t+1}/c_t)^{-\gamma}\ (1+R_{i,t+1}) - 1\}] = 0 \qquad i=1,\ldots, N;\ j=1,\ldots, L.$$

Now we have a lot of moments: LxN!

To estimate the model, we work with sample analogues of the moments:

$$g(w_t;\theta) = (1/T)\Sigma_t[Z_j\ \{\beta(c_{t+1}/c_t)^{-\gamma}\ (1+R_{i,t+1}) - 1\}] = 0$$

• Q: How do we choose $\mathbf{Z}_t$ the L instruments? Not a trivial question. In general, predetermined regressors are fine.

• Note: In the IV literature there is a big issue: weak instruments. In theory, we only need small correlation between **Z**t and the model's variables. However, the bigger the correlation, the better:

=> 1,000 weak instruments are no substitute for a strong instrument!

• Advantages of GMM approach:
- All we need is a moment condition.
- No need to log-linearize anything.
- Non-linearities are not a problem.
- Robust to heteroscedasticiy and distributional assumptions.

• Practical Considerations:
- We need at least as many moment conditions as parameters (just-identified case).
- If there are more moments –as it is usually the case-, we have "over-identifying restrictions." They can be used to test the model (Hansen's J- test):

$$J = T\, g(w_t;\theta)'\, S^{-1}\, g(w_t;\theta) \sim \chi^2_{LxN-k}$$

where $S = Var[g(w_t;\theta)]$

- Too many moments are not desirable in practice.
- The instruments (conditioning information) matter.

- Estimating S is tricky. In general, the moments will be serially dependent. Newey-West (1987) does not work well when the dimensions of the system is large. Small changes to S produces big swings in estimated θ. (Sometimes is better to work with W=I!)

- Some questions regarding the small sample properties of GMM.

• More practical considerations: Hansen's J-test

- The over-identifying restrictions are subject to a "which moments to choose?" critique.
- The J test also depends crucially on S; which cannot be estimated accurately.
- Not surprisingly, the J test rejects a lot of models. We should be aware of its problems.

• *Example: Hansen and Singleton (1982)*

For each asset i, H&S have:

$$E_t[Z_t\ \beta(c_{t+1}/c_t)^{-\gamma}\ (1+R_{i,t+1}) - 1] = 0 \qquad i=1,\ldots, N.$$

R_t = NYSE stock returns (VW and EW).
c_t = Consumption (Non-durables (ND) and ND plus services (NDS).)
Z_t= lagged R_{t+1} and c_{t+1}/c_t. (H&S use 1, 2, 4 and 6 lags.)

Findings: β close to 1 (around .99) and γ small between .32 to .03.
J-tests reject C-CAPM.

• A general problem with IV estimation in the C-CAPM: weak instruments. It's difficult to find instruments highly correlated with consumption growth.

•According to Hall's (1978) consumption follows a random walk:
lagged R_{t+1} and c_{t+1}/c_t should have low correlation with c_{t+1}/c_t!

• Nelson and Startz (1990): asymptotic theory can be a poor approximation in finite samples in the presence of weak instruments.
=> a true H_0 may be rejected. (The J test usually rejects C-CAPM.)

More C-CAPM Tests

- Mankiw and Shapiro (1986): Regress the average returns of the 464 surviving NYSE stocks (1959-1982) on their market β, on consumption growth betas, and on both betas to explain the cross section of average returns. Market β drive out consumption betas in multiple regressions.
- Breeden, Gibbons, and Litzenberger (1989): Work with industry and bond portfolios. CAPM and C-CAPM (with a "mimicking" portfolio = "maximum correlation porftolio" for consumption growth as the single factor) perform similarly. (Both rejected.)
- Cochrane (1996): Traditional CAPM substantially outperforms the canonical consumption-based model in pricing-size portfolios. For example, CAPM's root mean square pricing error (alpha) is 0.094 percent per quarter, while C-CAPM's is 0.54 percent per quarter.

Scaled Models

- Scaling = Conditioning Information.
- Since it adds information to models, usually it helps models (though, you may end up introducing redundant variables. Efficiency loss.)
- Scaling allows to have time-varying coefficients (recall conditional CAPM.)
- Go back to Euler's equation: $1 = E_t[m_{t+1}(1+R_{i,t+1})]$

Recall that any m_{t+1} satisfying the Euler's equation is an SDF candidate.

- Let $m_{t+1} = a_t + b_t R_{e,t+1}$ $R_{e,t+1}$ = return on market portfolio
  a_t and b_t to be found from Euler's equation
- We call models of the above form: *conditional linear factor models.*

- Substitute m_{t+1} in Euler's equation:

$$1 = E_t[(1+R_{i,t+1})]\ [1/(1+r_f)] + Cov_t[(1+R_{i,t+1}),\ a_t + b_t\ R_{e,t+1}]$$

$$E_t[(1+R_{i,t+1})] = (1+r_f) - Cov_t[(1+R_{i,t+1}),\ a_t + b_t\ R_{e,t+1}]\ (1+r_f)$$

$$E_t[R_{i,t+1}] = r_f - b_t\ Cov_t[R_{i,t+1},R_{e,t+1}]\ (1+r_f)$$

We have a conditional beta representation given by:

$$E_t[R_{i,t+1}] = r_f - b_t\ Var_t[R_{e,t+1}]\ (1+r_f)\ \beta_{i,t}$$

where

$$b_t = - (E_t[R_{e,t+1}] - r_f\ ) / Var_t[R_{e,t+1}]\ (1+r_f)$$

- If conditional moments are time-varying (and linear), b_t in the SDF will not be constant. Sources of variation:
  - $Var_t[R_{e,t+1}]$: Predictable volatility changes (very likely in HF data.)
  - r_f : The risk-free rate (though, it does not change a lot.)
  - $E_t[R_{e,t+1}]$: Forecastable excess returns (Recall predictability literature.)

- The scaling literature uses the forecasting instruments to (ad-hoc) model b_t.

  => Great source of papers: the formulation of b_t and a_t is ad-hoc.

- Example: Constructing a s*caled multifactor model*

(1) Define instruments: z_t is a forecasting variable for $E_t[R_{e,t+1}]$ –i.e., D/P (Campbell and Shiller (1988)), CAY (LL (2001), etc.

(2) Define a_t and b_t: Let a_t and b_t be linear functions of z_t:
$a_t = \gamma_0 + \gamma_1\ z_t$ and $b_t = \eta_0 + \eta_1\ z_t$

(3) Introduce a_t and b_t into m_{t+1}
$m_{t+1} = a_t + b_t\ R_{e,t+1} = \gamma_0 + \gamma_1\ z_t + (\eta_0 + \eta_1\ z_t)\ R_{e,t+1}$

(4) Generate multifactor model: Use Euler's equation, $E_t[m_{t+1}(1+R_{i,t+1})]$

$$1 = E_t[[\gamma_0 + \gamma_1\ z_t + \eta_0\ R_{e,t+1} + \eta_1\ (z_t\ R_{e,t+1})]\ (1+R_{i,t+1})]$$

Now, we have a 3-factor model!

The Puzzles

- The C-CAPM leads to three puzzles:
  - Equity Premium Puzzle -- Mehra and Prescott, (1985)
  - Risk Free Rate Puzzle -- Weil (1989)
  - Stock Market Volatility Puzzle -- Shiller (1982)

The Equity Premium Puzzle

• We can estimate an equity risk-premium using (CC.4):

$$E_t\,[r_{t+1}] - r_f = \gamma\sigma_{r\Delta} - \sigma_r^2/2$$

Actual difference: 4.18%
Average $[\ln(c_{t+1}) - \ln(c_t)] = .018$
Average $\sigma_r = .1674$
Estimated $\sigma_{r\Delta} = .0029$
Assume $\gamma = 19$ --too big for Mehra and Prescott (1985)

$$\Rightarrow E_t[r_{t+1}] - r_f = 19*.0029-.5\,(.1674)^2 = .04108 \text{ (log return!)}$$

• The calculations are an approximation, since the moments in (CC.4) are in terms of innovations. CLM say: Not a bad the approximation.

• Mehra and Prescott (1985) think that the highest plausible value for γ is 10. Then,

$$E_t\,[r_{t+1}] - r_f = .015,$$

which is very small relative to the observed risk premium 4.18% (estimated with over 100 years of data!).

This is the *equity risk-premium puzzle.*

Note: A large risk aversion coefficient, γ, is needed to resolve the puzzle. We need to amplify the low variability of consumption (or covariance with r_t).

Aside: Geometric vs. Arithmetic Average

• Technical issue, but it is first-order important in practice.

• Simple stock returns have fat right tails and truncated left tails (limited liability).
- Arithmetic averages are pulled to the right and exceeds the median.
- Geometric averages are close to the median.

• The difference between the two is about one-half the variance of returns, or about 1.5% for short-term returns.

Note: To the extent that stocks are mean-reverting, variance and arithmetic average decline with the holding period.

• **Current equity risk premium estimates**
• Dimson, Marsh, and Staunton (2006): 1900-2005 geometric average:
- World: 4.7%
- U.S.: 5.5%
- U.K.: 4.4%

• Cogley and Sargent (2008): U.S. numbers change over time (SD)
1872 – 2002: 4.10% (SD=17.34%)
1872 – 1928: 2.66% (SD=15.07%)
1929 – 1965: 7.08% (SD=22.39%)
1966 – 2002: 3.34% (SD=14.74%) --likely smaller after 08 crisis!

• Barro (2005): arithmetic (Real stock returns – Real Gov. bill returns):
- Japan (1923-2004): 9.2% - (-1.2%) = 10.4% (SD=27.1%)
- Canada (1934-2004): 7.4% - 1.0% = 6.3% (SD=16.3%)
- U.S. (1880-2004): 8.1% - 1.5% = 6.6% (SD=19.1%)
- U.K. (1880-2004): 6.3% - 1.6% = 4.7% (SD=17.9%)
- France (1896-2004): 7.0% - (-1.8%) = 8.8% (SD=27.9%)

The Risk-free Puzzle

• We can estimate γ or β –given the other- using the formulation for the risk-free rate:

$$r_f = -\ln\beta + \gamma\, E_t[\ln(c_{t+1}) - \ln(c_t)] - (1/2)\,\gamma^2\,\sigma_\Delta^2 \qquad \text{(CC.5)}$$

Average $r_f = .018$
Average $[\ln(c_{t+1}) - \ln(c_t)] = .018$
Average $\sigma_\Delta = .0328$
Assume $\gamma = 19$ => β=1.12 > 1, a negative rate of time preference!

Weil (1989) calls this the *risk-free puzzle*.

(CC.5) presents r_f as a quadratic function of γ. The last term in (CC.5) is called the "precautionary savings effect." But, it is usually ignored since σ_Δ is low. Thus, economists (Weil, among them) think of a positive relation between r_f and γ.

• Siegel (1999) points out that the low returns on fixed interest are the "puzzle."

Siegel observes that real equity returns around 7% have been stable over time and justifiable (possible survivorship bias, limited diversification, portfolio management costs, etc.)

Hansen-Jagannathan Bounds

- Consumption based models don't work empirically – equity premium puzzle. Instead of just trying a bunch of different utility functions, it is helpful to characterize some properties that m_{t+1} must satisfy.
- HJ bounds – bound on $\{\sigma(m_{t+1}), E(m_{t+1})$, other moments of $m_{t+1}\}$
- Purpose:

  (1) Give us a clearer understanding of why certain asset pricing models are rejected by the data.

  (2) Allow us to compare asset pricing models against one another.

  (3) Help to identify features of the data that present the most stringent restrictions on asset pricing models.

- What is an asset pricing model?

• HJ bound using a single return.

From the Euler's equation:

$$1= E_t[m_{t+1} (R_{t+1}-r_f)]$$

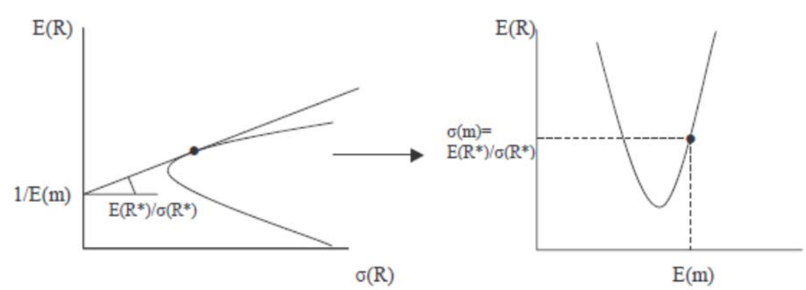
Taking logs: $0 = E_t[m_{t+1}]\, E_t[y_{t+1}] + \rho\, \sigma_y\, \sigma_m$
where
$y_{t+1} = R_{t+1}-r_f$
$\rho = Cov_t(m_{t+1},y_{t+1})$

Then, $\sigma_m = - E_t[m_{t+1}]\ \{E_t[y_{t+1}] /(\rho\, \sigma_y)\}$ $(\rho \neq 0)$

Since ρ is between [-1,1] => $\sigma_m \geq E_t[m_{t+1}]\ \{E_t[y_{t+1}] /\sigma_y \}$

$$\sigma_m/E_t[m_{t+1}] \geq E_t[(R_{t+1}-r_f)] /\sigma_R \qquad \text{(=Sharpe ratio)}$$

• Theorem (HJ Bounds): $\sigma_m/E_t[m_{t+1}]$ must be at least as large as the maximum SR attained by any portfolio.



• Equity premium puzzle (again):

$$\sigma_m = \sigma(u'_{t+1}/ u'_t) \geq (1/(1+r_f))\ |E_t[(R_{t+1}-r_f)] /\sigma_R|$$

For the power utility model: $\sigma_m = \sigma(\beta(c_{t+1}/c_t)^{-\gamma}) \geq (1/r_f)\ |E_t[(R_{t+1}-r_f)] /\sigma_R|$

• Observed SR of stock market indices is too high (.06/.18=.33), relative to (low) the volatility of consumption (.033) => (unrealistically) high level of risk aversion

• HJ bound using a vector of returns (no restrictions $m_{t+1} \geq 0$)

Start with Euler's equation:

$$\mathbf{p_t} = E_t[\mathbf{m}_{t+1}\ \mathbf{x}_{t+1}] \qquad \text{(Nx1 vectors)}$$

Think of regressing m_{t+1} on x_{t+1}:

$\mathbf{m}_{t+1} = E_t[\mathbf{m}_{t+1}] + (\mathbf{x}_{t+1} - E_t[\mathbf{x}_{t+1}])' \boldsymbol{\delta} + \varepsilon_{t+1}$ --ε_{t+1} iid ~ $D(0,\text{Var}_t(\varepsilon_{t+1}))$.

Note that $\text{Var}(\mathbf{m}_{t+1}) = \boldsymbol{\delta}' \text{Var}_t(\mathbf{x}_{t+1})\ \boldsymbol{\delta} + \text{Var}_t(\varepsilon_{t+1})$.

From the Euler's condition,

$$\begin{aligned}
\mathbf{p}_t &= E_t[\mathbf{x}_{t+1}]\ E_t[\mathbf{m}_{t+1}] + \text{Cov}_t[\mathbf{x}_{t+1}, \mathbf{m}_{t+1}] \\
&= E_t[\mathbf{x}_{t+1}]\ E_t[\mathbf{m}_{t+1}] + \text{Cov}_t[\mathbf{x}_{t+1}, E_t[\mathbf{m}_{t+1}] + (\mathbf{x}_{t+1} - E_t[\mathbf{x}_{t+1}])' \boldsymbol{\delta} + \varepsilon_{t+1}] \\
&= E_t[\mathbf{x}_{t+1}]\ E_t[\mathbf{m}_{t+1}] + \text{Cov}_t[\mathbf{x}_{t+1}, \mathbf{x}_{t+1}']\ \boldsymbol{\delta} \\
&= E_t[\mathbf{x}_{t+1}]\ E_t[\mathbf{m}_{t+1}] + \text{Var}_t(\mathbf{x}_{t+1})\ \boldsymbol{\delta} \\
&\Rightarrow \boldsymbol{\delta} = \{\text{Var}_t(\mathbf{x}_{t+1})\}^{-1} \{\mathbf{p}_t - E_t[\mathbf{x}_{t+1}]\ E_t[\mathbf{m}_{t+1}]\}
\end{aligned}$$

$$\Rightarrow \boldsymbol{\delta} = \{\text{Var}_t(\mathbf{x}_{t+1})\}^{-1} \{\mathbf{p}_t - E_t[\mathbf{x}_{t+1}]\ E_t[\mathbf{m}_{t+1}]\}$$

Recall that $\text{Var}(\mathbf{m}_{t+1}) = \boldsymbol{\delta}' \text{Var}_t(\mathbf{x}_{t+1})\ \boldsymbol{\delta} + \text{Var}_t(\varepsilon_{t+1})$. Then,

$$\text{Var}(\mathbf{m}_{t+1}) \geq \{\mathbf{p}_t - E_t[\mathbf{x}_{t+1}]\ E_t[\mathbf{m}_{t+1}]\}\ \{\text{Var}_t(\mathbf{x}_{t+1})\}^{-1} \{\mathbf{p}_t - E_t[\mathbf{x}_{t+1}]\ E_t[\mathbf{m}_{t+1}]\}$$

This is an hyperbola in $\{E_t[\mathbf{m}_{t+1}], \text{Var}(\mathbf{m}_{t+1})\}$ space.

As we go through values of $E_t[\mathbf{m}_{t+1}]$, from higher to lower, the slope to the tangency portfolio on the efficient frontier falls until $1/E_t[\mathbf{m}_{t+1}]$, equals the expected return on the minimum variance portfolio. As $E_t[\mathbf{m}_{t+1}]$ falls further, the SR increases.

• Bounds on other moments of m_{t+1} can also be found. If risk-free asset exist, we can compute bounds with the restriction that $m_{t+1} > 0$.

• The m_{t+1} on the HJ bound is perfectly negatively correlated with the excess return of the tangency portfolio.

• Using the HJ bound to rule out asset pricing models

• Suppose we assume the power utility model presented above:

$$m_{t+1} = \beta(c_{t+1}/c_t)^{-\gamma}$$

1. Would γ=0 be a good model? $m_{t+1} = \beta$ => $\sigma_m = 0$. No!

2. γ=1 (log utility)? $m_{t+1} = \beta\ (c_t/c_{t+1})$ => $\sigma_m = 1\%$. No!

3. We need high γ for σ_m not to violate the HJ bounds.

• Q: Does adding an asset class expand the efficient frontier? Same as asking if adding these assets cause the HJ bounds to go down.

• The HJ are based on point estimates, which all are measured with error. Burnside (1994) proposes a number of tests to evaluate the HJ bounds. The vertical distance, h, test is based on:

$$h = \sigma(\mathbf{m}_{t+1}) - [\{\mathbf{p}_t - E_t[\mathbf{x}_{t+1}]\ E_t[\mathbf{m}_{t+1}]\}\{\mathrm{Var}_t(\mathbf{x}_{t+1})\}^{-1}\{\mathbf{p}_t - E_t[\mathbf{x}_{t+1}]\ E_t[\mathbf{m}_{t+1}]\}]^{1/2}$$

Replacing by the sample counterparts, we get $\hat{h}$. $\hat{h}$, once appropriately scaled by $\sigma(\hat{h})$ converges in distribution to a standard normal.

The Excess Volatility Puzzle

• Asset prices volatility is too high to be explained by fundamentals –i.e., earnings and dividends. Shiller (1982) and LeRoy and Porter (1981).

• Both papers have serious econometric (time-series) issues.

• Shiller (1982) pointed out this by looking at the ex-post NPV of dividends and computing stock market theoretical volatility.

$P_t = E_t[(P_{t+1} + D_{t+1})/\zeta_t]$ where ζ_t is the discount factor.

Repeated substitution (assuming $\zeta_t = \zeta$ –i.e., constant discount factor):

$$P_t = E_t[\Sigma_k D_{t+k}/\zeta^k] + E_t[P_{t+K}/\zeta^K] =$$
$$= E_t[\Sigma_k D_{t+k}/\zeta^k] = P_t^* \quad \text{(impose transversality condition)}$$

• Shiller (1982) computed the volatility of P_t and P_t^*.

• We do not know P_t^*. But, ex-post we can compute it. The basic assumption behind the comparison is rationality:

$$P_t = E_t[P_t^*].$$

Then,

$$P_t^* = P_t + \varepsilon_t.$$

$$Var(P_t^*) = Var[P_t] + Var[\varepsilon_t] \quad \Rightarrow Var(P_t^*) > Var[P_t].$$

But, Shiller (1982) found $Var(P_t^*) < Var[P_t]$ (a big difference!)

Problems:
(1) No bubbles allowed in solution/imposition of $P_T = P_T^*$.
(2) NPV calculations are good for risk-neutrality.
(3) Ex-post = ex-ante.
(4) Unit roots, serial correlation.
(5) Finite sample manipulation of D_t (endogeneity issue).

• Mehra and Prescott (1985) changes the center of attention to the equity premium. But, the HJ bounds brings it back.

Another C-CAPM inconsistency

- We can use (CC.5) to estimate γ (let's assume β=.995):

$$r_f = - \ln \beta + \gamma\, E_t[\ln(c_{t+1}) - \ln(c_t)] - (1/2)\, \gamma^2\, \sigma_\Delta^2$$

- Or we can use (CC.4) : $E_t\,[r_{t+1}] – r_f = \gamma\sigma_{r\Delta} - \sigma_r^2/2$

Problem: In practice, the estimates of γ in both regressions are in total disagreement. Not a good result for the model.